The Barcan Formula and Our Talk of Possible Worlds

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Possible non-Writers

"But to say that *some people possibly are not writers* is modally the same as saying that *possibly some people are not writers*, and although one implies the other the meaning of the one may be opposite to the other."

(see Williamson 2013, pg 45)

Ibn Sina

Logics in General

Language	**Proof Theory**	**Semantics**
$\mathbf{L}$	$\vdash$	$\models$

Logical Consequence

A *tree* is a partial order with a unique maximum element x_0, such that for any element x_n, there is a unique finite chain of elements $x_n \leq x_{n-1} \leq \cdots \leq x_1 \leq x_0$. The initial list of a tree is the sequence of a premises.

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We say A is a logical consequence of Σ ($\Sigma \vdash A$) if there exists a tree with the initial list as Σ and $\neg A$ which is closed.

What is Modal about Modal Logic?

Modal logics are logics which have extra unary propositional operators. The usual modal logics are *alethic* ($\Box, \Diamond$), *temporal* (G, H, F, P), *epistemic* (K) and *deontic* (O, P). We are essentially, adding one more parameter to keep track of.

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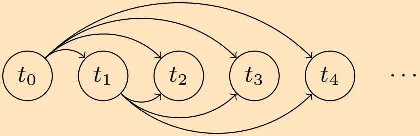

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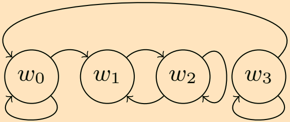

De Re and *De Dicto*

’The tallest building will always be Asian.’ is ambiguous.

$$\Box \exists x A(x)$$

$$\exists x \Box A(x)$$

Quantified Modal Logic

Q: How do we extend propositional modal logic to quantified modal logic?

Quantified Modal Logic

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A: Just put them together!

Quantified Modal Logic

$$\neg(A \supset B) \text{ - i}$$
$$|$$
$$A \text{ - i}$$
$$|$$
$$\neg B \text{ - i}$$

$$A \supset B \text{ - i}$$
$$\swarrow \quad \searrow$$
$$\neg A \text{ - i} \qquad B \text{ - i}$$

$$\neg\neg P \text{ - i}$$
$$|$$
$$P \text{ - i}$$

Quantified Modal Logic

$$\begin{array}{cccc} \neg\Box P \text{ - i} & \neg\Diamond P \text{ - i} & \Box P \text{ - i iRj} & \Diamond P \text{ - i} \\ \Big| & \Big| & \Big| & \Big| \\ \Diamond\neg P \text{ - i} & \Box\neg P \text{ - i} & P \text{ - j} & \text{iRj } P \text{ - j} \end{array}$$

Quantified Modal Logic

$$
\begin{array}{cccc}
\neg\exists x P \text{ - i} & \neg\forall x P \text{ - i} & \forall x P \text{ - i} & \exists x P \text{ - i} \\
| & | & | & | \\
\forall x \neg P \text{ - i} & \exists x \neg P \text{ - i} & P_x(a) \text{ - i} & P_x(c) \text{ - i}
\end{array}
$$

The Barcan Formula

Ruth Barcan Marcus

$$\neg(\Diamond\exists x P(x) \supset \exists x \Diamond P(x)) \text{ - } 0$$

$$\Diamond\exists x P(x) \text{ - } 0$$

$$\neg\exists x \Diamond P(x) \text{ - } 0$$

$$\exists x P(x) \text{ - } 1$$

$$P(a) \text{ - } 1$$

$$\neg\Diamond P(a) \text{ - } 0$$

$$\neg P(a) \text{ - } 1$$

Hence, we have $\Diamond\exists x P(x) \supset \exists x \Diamond P(x)$

Constant Domain Semantics

An interpretation of constant domain semantics is a quadrupule $\langle D, W, R, v \rangle$.

W is the set of possible worlds, R is the accessibility relation, D is the domain of quantification and v assigns the elements of the language to appropriate elements or subsets of D.

Varying Domain Semantics

An interpretation of variable domain semantics is also a quadrupule $\langle D, W, R, v \rangle$.

W is the set of possible worlds, R is the accessibility relation, D is the domain of all objects (not quantification!) and v assigns the elements of the language to appropriate elements or subsets of D. In addition to this, v assigns each $w \in W$ a subset of D, which is the domain of quantification for that world.

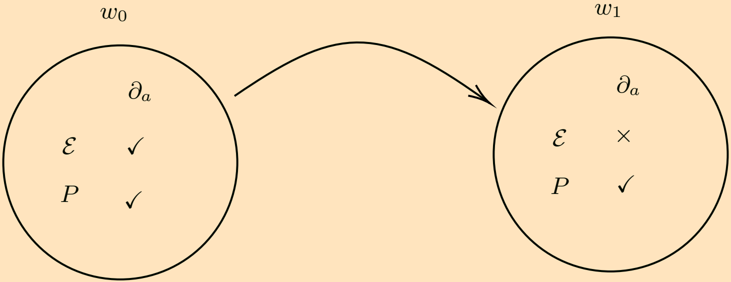
Varying Domain Semantics

Now consider $\forall x P(x)$ holds at world w. Then, for every object in the domain of quantification of the world, a , $P(a)$ holds. Now, say $b \notin v(w)$. Then it is not the case that $P(b)$ holds. Universal instantiation rule fails. Instead, we will add a special unary predicate $\mathcal{E}$ as is done in free logic.

Varying Domain Semantics

$$\begin{tikzcd}
 & & & \exists x P \text{ - i} \arrow[d, dash] \\
 & \forall x P \text{ - i} \arrow[dl, dash] \arrow[dr, dash] & & \mathcal{E} c \text{ - i} \arrow[d, dash] \\
\neg \mathcal{E} a \text{ - i} & & P_x(a) \text{ - i} & P_x(c) \text{ - i}
\end{tikzcd}$$

Counter-Model in Varying Domain Semantics



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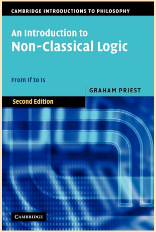

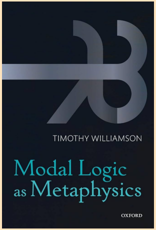

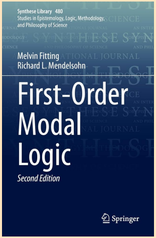

Semantic Consequence

An interpretation of the language **L** is $\mathcal{T} = (D, W, R, v)$. D is a non-empty set and v is a function such that $v(c) \in D$ and $v_w(R_i) \subseteq D^n$ if R_i is n-ary. For every element d in D, add a constant k_d to **L**.

The compatibility conditions are:

- $v(R_i(a_1, \cdots, a_n)) = 1$ iff $(v(a_1), \cdots, v(a_n)) \in v_w(R_i)$
- $v(\forall x A) = 1$ iff for all $d \in D$, $v(A(k_d/x)) = 1$
- $v(\exists x A) = 1$ iff for some $d \in D$, $v(A(k_d/x)) = 1$
- v follows the same compatibility conditions for propositional languages.

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- v follows the same compatibility conditions for propositional languages.

A sentence Q is a *semantic consequence* of a set of sentences Σ iff there is no interpretation v, such that $v(P) \in \mathcal{D}$ for all $P \in \Sigma$ but $v(Q) \notin \mathcal{D}$. In case Q is a semantic consequence of Σ, we write $\Sigma \vDash Q$.
A tautology A is a sentence such that it is a semantic consequence of the empty set, i.e., $\vDash A$.

Completeness and Soundness

Soundness of validity and entailment: If $\Sigma \vdash Q$, then $\Sigma \models Q$.

Completeness of validity and entailment: If $\Sigma \models Q$, then $\Sigma \vdash Q$.

Theorem: The system **K** is sound and complete with respect to the constant domain semantics and the constant domain tableaux.

Theorem: The system **K** is sound and complete with respect to the variable domain semantics and the variable domain tableaux.

Monotonicity and Barcan Formulae

Theorem: Let $\mathcal{F} = \langle W, R, D \rangle$ be a varying domain frame. The following are equivalent:

- $\mathcal{F}$ is monotonic, i.e., for all w, v such that vRw, $D_v \subseteq D_w$
- The Converse Barcan Formula is valid in every model based on $\mathcal{F}$.
- $\mathcal{E}(x) \supset \Box\mathcal{E}(x)$ is valid in every normal model based on $\mathcal{F}$.
- $\forall x \Box \mathcal{E}(x)$ is valid in every normal model based on $\mathcal{F}$.

Theorem: Let $\mathcal{F} = \langle W, R, D \rangle$ be a varying domain frame. The following are equivalent:

- $\mathcal{F}$ is anti-monotonic, i.e., for all w, v such that vRw, $D_w \subseteq D_v$
- The Barcan Formula is valid in every model based on $\mathcal{F}$.
- $\Diamond\mathcal{E}(x) \supset \mathcal{E}(x)$ is valid in every normal model based on $\mathcal{F}$.